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Finite-volume corrections to the mean-field solution of the SK model

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Abstract. This paper shows in a simple way how non-integer exponents can arise in the study of the finite-N corrections to the free energy density of the mean-field solution of the SK model of spin glasses. The well known mean-field solution obtained with the hierarchical ansatz is valid in the thermodynamical limit $N \to \infty$. We study the fluctuations associated with the longitudinal eigenvalues of the free-energy Hessian in the neighbourhood of the critical point, neglecting the interactions between different modes. We find that the contribution of the positive eigenvalues is of the order of $N^{-3/4} \ln N$; for the zero modes, we find that they cannot be treated separately from the ones coming from the non-longitudinal fluctuations because they would give a divergent contribution.

1. Introduction

The study of spin glasses [1] and, in general, of frustrated systems in the last 15 years has developed a very strong calculation tool—the replica method—which has found a huge number of applications in the physics of disordered systems [2].

The Sherrington–Kirkpatrick (SK) model [3] is certainly the most widely studied model of spin glass but up to now, from the theoretical point of view, it has always been treated in the thermodynamical ($N \to \infty$) limit.

In this paper we execute the first step of the $1/N$ expansion around the mean-field solution obtained with the hierarchical ansatz; in section 2 we introduce the model and the mean-field solution; in section 3, we show our expansion; next, we define four critical exponents in terms of which we can obtain the first correction to the mean-field solution and finally, in section 5, we show the results obtained for the critical exponents and for the free energy density.

Our calculation are based upon two approximations: the first one is to consider only longitudinal fluctuations of the order parameter (see section 3). This is a good assumption, because a complete study [4] shows that the dominant fluctuations are precisely the longitudinal ones. The major problem that we face treating only these eigenvalues is that the zero modes give a divergent contribution. The reason for this

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divergence is that their contribution is cancelled by that of the zero modes of the transverse fluctuations which we do not include in this approximation.

Next, we will neglect the interactions between the different modes—this is nothing but the Gaussian approximation. We will use it in (3.15). This approximation prevents us from finding the exact numerical values of the exponents in the finite-N correction, but our task is to show that non-integer power laws and logarithmic dependences arise from the beginning in the $1/N$ expansion. The values presented in the final section must then be regarded as a first approximation to the real numerical values.

2. The SK model: mean-field solution

The SK model [3] is a magnetic system composed by N Ising spins $\{S_i = \pm 1,\ i = 1, \ldots, N\}$, with a Hamiltonian

$$\mathcal{H}_{[J]}\{S\} = -\frac{1}{2}\sum_{i \neq j} J_{ij} S_i S_j - h \sum_i S_i \tag{2.1}$$

where the parameters $\{J_{ij}\}$ are chosen at random from the distribution

$$\mathrm{P}(J_{ij}) = \sqrt{\frac{N}{2\pi J^2}} \exp\left(-\frac{N J_{ij}^2}{2J^2}\right). \tag{2.2}$$

The replica solution gives the free energy density of the SK model, defined as the average over the probability distribution of the J's of the quantity

$$f_{[J]} \equiv -\frac{1}{N\beta} \ln \sum_{\{S\}} \exp\left(-\beta \mathcal{H}_{[J]}\{S\}\right). \tag{2.3}$$

It can be proven [3, 5] that

$$f \equiv \int \cdots \int \left[\prod_{i<j} \mathrm{d}J_{ij} \mathrm{P}(J_{ij})\right] f_{[J]} \tag{2.4}$$

is given by

$$f = -\frac{1}{\beta} \lim_{n \to 0^+} \left\{ \frac{\beta^2 J^2}{4} \left(1 - \frac{2}{n} \sum_{a<b} Q_{ab}^2\right) + \frac{1}{n} \ln \mathrm{Tr}_n \exp \mathrm{L}[Q] \right\} \tag{2.5}$$

with

$$\mathrm{L}[Q] = \beta^2 J^2 \sum_{a<b} Q_{ab} S^a S^b. \tag{2.6}$$

Tr_n denotes the sum over all the possible configurations of n Ising spins $\{S^a,\ a = 1, \ldots, n\}$.

For integer n, (Q_{ab}) is an $n \times n$ matrix, the elements of which must minimize the free energy density in the $n \to 0^+$ limit.

The hierarchical ansatz for the replica solution [6–8] gives the parametrization for integer n that supplies the correct solution of the SK model. The expression of the replica matrix Q for integer n is obtained as the $K \to \infty$ limit of block matrices $Q^{(K)}$, defined in terms of the two families of parameters, $\{q_i \in \mathbb{R},\ i = 1, \ldots, K\}$ and $\{m_j \in \mathbb{N},\ j = 1, \ldots, K+1\}$, by

$$Q_{ab} = q_i \qquad \text{if} \quad \begin{cases} \mathrm{Int}\left(\dfrac{a}{m_i}\right) \neq \mathrm{Int}\left(\dfrac{b}{m_i}\right) \\ \mathrm{Int}\left(\dfrac{a}{m_{i+1}}\right) = \mathrm{Int}\left(\dfrac{b}{m_{i+1}}\right). \end{cases} \tag{2.7}$$

In the $n \to 0^+$ limit, the two sequences of parameters $\{q_i\}$ and $\{m_j\}$—that supply the dimension of the jth block in the matrix $Q^{(K)}$—can be unified defining a function

$$q(x) = q_i \qquad \text{when} \quad m_i \leqslant x < m_{i+1} \tag{2.8}$$

that, in the $K \to \infty$, $n \to 0^+$ limit, becomes a continuous function in the interval $x \in [0, 1]$.

With this ansatz, the expression for the free energy density vaild near the spin glass transition is [8]

$$-\beta f = \ln 2 + \frac{\beta^2 J^2}{4} + \frac{1}{N} \ln \max_{[q]} \exp\left(-N\mathcal{F}[q]\right) \tag{2.9}$$

with

$$\mathcal{F}[q] = \frac{1}{2}\int_0^1 \mathrm{d}x \left(\tau q(x)^2 - \frac{x\, q(x)^3}{3} - q(x)\int_0^x q(y)^2 \mathrm{d}y + \frac{q(x)^4}{4}\right) \tag{2.10}$$

where $2\tau = 1 - T^2/T_c^2$ and $T_c = J$.

The order parameter function that minimizes the free energy is

$$q_{\mathrm{CM}}(x) = \begin{cases} x/3 & \text{if} \quad 0 \leqslant x \leqslant 3q_1 \\ q_1 & \text{if} \quad 3q_1 \leqslant x \leqslant 1 \end{cases} \tag{2.11}$$

with $q_1 = t + t^2 + \mathcal{O}(t^3)$ and $t = 1 - T/T_c$.

In the following sections we will estimate the order of the first correction to this solution due to finite-volume effects.

3. Finite-N corrections

Now that we have the right solution in the limit $N \to \infty$, we must face the problem of calculating the first finite-volume corrections. The results will be particularly interesting for the comparison between numerical simulations (made obviously with a finite number of spins) and theoretical predicitons.

As the starting point for the $1/N$ expansion we consider the approximate expression (2.10) for the free energy density, valid near the critical point; to perform the expansion in all the spin-glass phases we should know the eigenvalues of the complete Hessian of the free energy for all the values of T and h.

At this point we will make use of the first approximation presented in the introduction: we consider only the longitudinal fluctuations of the $\{Q_{ab}\}$ matrix around the mean-field solution, i.e. those which preserve the replica symmetry-breaking scheme introduced in the hierarchical ansatz. This approximation is equivalent to studying only the fluctuations of the order parameter function $q(x)$ around its saddle point solution $q_{\rm CM}(x)$.

We then put $q(x) = q_{\rm CM}(x) + \delta q(x)$ in (2.10) and obtain

$$\mathcal{F}[q] = \mathcal{F}_0 + \delta_2\mathcal{F}[\delta q] + \delta_3\mathcal{F}[\delta q] + \delta_4\mathcal{F}[\delta q] \tag{3.1}$$

with

$$\mathcal{F}_0 = \frac{1}{2}(\tau q_1^3 - q_1^5) = \frac{t^4}{2}\left(1 - \frac{t}{2} + \mathcal{O}(t^2)\right) \tag{3.2}$$

$$\delta_2\mathcal{F}[\delta q] = \frac{1}{2}\int_0^1 \mathrm{d}x \left(\tau\delta q(x)^2 - x\, q_{\rm CM}(x)\delta q(x)^2 - q_{\rm CM}(x)\int_0^x \delta q(y)^2 \mathrm{d}y + \right.$$
$$\left. -2\delta q(x)\int_0^x q_{\rm CM}(y)\,\delta q(y)\,\mathrm{d}y + \frac{3}{2}q_{\rm CM}(x)^2\delta q(x)^2\right) \tag{3.3}$$

$$\delta_3\mathcal{F}[\delta q] = \frac{1}{2}\int_0^1 \mathrm{d}x \left(-\frac{x\,\delta q(x)^3}{3} - \delta q(x)\int_0^x \delta q(y)^2\mathrm{d}y + q_{\rm CM}(x)\delta q(x)^3\right) \tag{3.4}$$

$$\delta_4\mathcal{F}[\delta q] = \frac{1}{2}\int_0^1 \mathrm{d}x\, \frac{\delta q(x)^4}{4}\,. \tag{3.5}$$

Following [9], we diagonalize the quadratic part looking for normalized functions $f_\lambda(x)$ that verify the equation

$$2\int_0^x q_{\rm CM}(y)\, f_\lambda(y)\,\mathrm{d}y + 2q_{\rm CM}(x)\int_0^x f_\lambda(y)\,\mathrm{d}y = \lambda\, f_\lambda(x) \tag{3.6}$$

because for these we have

$$\delta_2\mathcal{F}[f_\lambda] = \lambda\int_0^1 f_\lambda(x)^2\,\mathrm{d}x = \lambda\,. \tag{3.7}$$

The solutions of (3.6) can be separated into two families: (1) $f_\lambda(x) \neq 0$ only if $0 \leqslant x \leqslant 3q_1$, (2) $f_\lambda(x) \neq 0$ only if $3q_1 \leqslant x \leqslant 1$.

In the first case, taking $f_\lambda(x) = a_\lambda \cos\omega_\lambda x + b_\lambda \sin\omega_\lambda x$, we have

$$\omega_\lambda = \sqrt{\frac{2}{3\lambda}} \qquad a_\lambda = 3\omega_\lambda b_\lambda \tag{3.8}$$

$$b_\lambda = 2\omega_\lambda\left[6\omega_\lambda + 3q_1\omega_\lambda + 27q_1\omega_\lambda^3 - 6\omega_\lambda\cos^2(3\omega_\lambda q_1)\right.$$
$$\left. + (9\omega_\lambda^2 - 1)\cos(3\omega_\lambda q_1)\sin(3\omega_\lambda q_1)\right]^{-1/2} \tag{3.9}$$

$$\frac{-a_\lambda \sin(3\omega_\lambda q_1) + b_\lambda\cos(3\omega_\lambda q_1)}{a_\lambda\cos(3\omega_\lambda q_1) + b_\lambda\sin(3\omega_\lambda q_1)} = \frac{2 - 6q_1}{3\omega_\lambda\lambda}\,. \tag{3.10}$$

The λ's that verify (3.10) are the eigenvalues of the Hessian; they come out positive and have an accumulation point in $\lambda = 0$.

In the second case, if $\lambda \neq 0$, (3.6) imposes that f_λ is constant ($f_\lambda = 1/\sqrt{1-3q_1}$), and in that case the resulting eigenvalue is $\lambda = 2q_1(1-3q_1)$. Otherwise, all the eigenfunctions have $\lambda = 0$, since in this case (3.6) reduces to

$$2q_1 \int_{3q_1}^{1} f_\lambda(y)\,\mathrm{d}y = 0\,. \tag{3.11}$$

Now that we have found the solutions of (3.6), we can study the expansion of (2.10). We begin by making the change of variable $\delta q(x) \to \delta q(x)/\sqrt{N}$:

$$-\beta f = \ln 2 + \frac{\beta^2 J^2}{4} - \mathcal{F}_0 + \frac{1}{N}\ln \iint \frac{\mathcal{D}\delta q(x)}{\sqrt{N}} \exp\left(-\delta_2\mathcal{F}[\delta q] - \frac{\delta_3\mathcal{F}[\delta q]}{\sqrt{N}} - \frac{\delta_4\mathcal{F}[\delta q]}{N}\right) \tag{3.12}$$

where by the double integration sign we mean the functional integral over all the $\delta q(x)$; now if we introduce a complete set of eigenfunctions $f_\lambda(x)$ with eigenvalues λ and expand $\delta q(x)$ in this basis, we will have

$$\delta q(x) = \sum_\lambda y_\lambda f_\lambda(x) \tag{3.13}$$

$$\begin{aligned} -\beta f = \ln 2 + \frac{\beta^2 J^2}{4} - \mathcal{F}_0 & \\ + \frac{1}{N}\ln \int \prod_\lambda \frac{\mathrm{d}y_\lambda}{\sqrt{N}} \exp\Bigg(& -\sum_\lambda \lambda y_\lambda^2 - \frac{1}{\sqrt{N}} \sum_{\lambda,\lambda',\lambda''} \mathcal{M}^{(3)}_{\lambda\lambda'\lambda''} y_\lambda y_{\lambda'} y_{\lambda''} \\ & - \frac{1}{N} \sum_{\lambda,\lambda',\lambda'',\lambda'''} \mathcal{M}^{(4)}_{\lambda\lambda'\lambda''\lambda'''} y_\lambda y_{\lambda'} y_{\lambda''} y_{\lambda'''} \Bigg)\,. \end{aligned} \tag{3.14}$$

Now we must estimate the matrices $\mathcal{M}^{(3)}$ and $\mathcal{M}^{(4)}$ to proceed with the $1/N$ expansion. Here we will introduce our second approximation. Since both matrices have elements that tend to zero when we go far from the diagonal, we introduce the following hypothesis:

$$\begin{aligned} \mathcal{M}^{(3)}_{\lambda\lambda'\lambda''} &= \delta_{\lambda\lambda'}\,\delta_{\lambda\lambda''}\, M^{(3)}_\lambda \\ \mathcal{M}^{(4)}_{\lambda\lambda'\lambda''\lambda'''} &= \delta_{\lambda\lambda'}\,\delta_{\lambda\lambda''}\,\delta_{\lambda\lambda'''}\, M^{(4)}_\lambda\,. \end{aligned} \tag{3.15}$$

This simplification is nothing but the Gaussian approximation for the fluctuations of our system. We are supposing that the fluctuations with different λ's are independent, so that the matrices $\mathcal{M}^{(3)}$ and $\mathcal{M}^{(4)}$ are diagonal.

If we substitute (3.15) in (3.14) we obtain

$$\begin{aligned} -\beta(f - f_0) &= \frac{1}{N}\ln \int \prod_\lambda \frac{\mathrm{d}y_\lambda}{\sqrt{N}} \exp\left[\sum_\lambda \left(-\lambda y_\lambda^2 - \frac{M^{(3)}_\lambda}{\sqrt{N}} y_\lambda^3 - \frac{M^{(4)}_\lambda}{N} y_\lambda^4\right)\right] \\ &= \frac{1}{N}\ln \left[\prod_\lambda \int \frac{\mathrm{d}y_\lambda}{\sqrt{N}} \exp\left(-\lambda y_\lambda^2 - \frac{M^{(3)}_\lambda}{\sqrt{N}} y_\lambda^3 - \frac{M^{(4)}_\lambda}{N} y_\lambda^4\right)\right] \\ &= \frac{1}{N}\sum_\lambda \left[\ln \int \frac{\mathrm{d}y_\lambda}{\sqrt{N}} \exp\left(-\lambda y_\lambda^2 - \frac{M^{(3)}_\lambda}{\sqrt{N}} y_\lambda^3 - \frac{M^{(4)}_\lambda}{N} y_\lambda^4\right)\right] \end{aligned} \tag{3.16}$$

where we have put $f_0 \equiv -1/\beta(\ln 2 + \beta^2 J^2/4 - \mathcal{F}_0)$.

When the number of spins is $N \gg 1$, the final sum in (3.16) is dominated by the small values of λ because the bigger values are cut by the quadratic part of the exponential. On the other hand, there are many such small values because $\lambda = 0$ is an accumulation point of the eigenvalues, so that it is convenient to substitute the sum over λ by an integral weighted with an eigenvalue density $\rho(\lambda)$, separating the contribution of the zero modes from that of the non-zero ones.

Now, for large values of λ, the integral in (3.16) is dominated by the quadratic part of the exponential, while for small eigenvalues the important terms are the cubic and the quartic ones. Let λ_{cr} be the separating value of λ between the two regions; dividing the contribution of the zero modes, the contribution of small λ and that of the bigger eigenvalues, the expression (3.16) for the free energy density becomes

$$
\begin{aligned}
-\beta(f - f_0) &= \frac{1}{N} \sum_{\lambda=0} \ln \frac{1}{\sqrt{N}} \int \mathrm{d}y \exp\left(-\frac{M_0^{(3)}}{\sqrt{N}} y^3 - \frac{M_0^{(4)}}{N} y^4 \right) \\
&\quad + \frac{1}{N} \int_0^{\lambda_{cr}} \mathrm{d}\lambda\, \rho(\lambda) \ln \frac{1}{\sqrt{N}} \int \mathrm{d}y \exp\left(-\frac{M_\lambda^{(3)}}{\sqrt{N}} y^3 - \frac{M_\lambda^{(4)}}{N} y^4 \right) \\
&\quad + \frac{1}{N} \int_{\lambda_{cr}} \mathrm{d}\lambda\, \rho(\lambda) \ln \frac{1}{\sqrt{N}} \int \mathrm{d}y\, \mathrm{e}^{-\lambda y^2} .
\end{aligned}
\tag{3.17}
$$

Let us consider separately the three terms of (3.17), beginning with the third and easiest one.

The Gaussian integral is easily computable:

$$
\frac{1}{N} \int_{\lambda_{cr}} \mathrm{d}\lambda\, \rho(\lambda) \ln \frac{1}{\sqrt{N}} \int_{-\infty}^{+\infty} \mathrm{d}y\, \mathrm{e}^{-\lambda y^2} = \frac{1}{2N} \int_{\lambda_{cr}} \mathrm{d}\lambda\, \rho(\lambda) \ln \frac{\pi}{\lambda N} . \tag{3.18}
$$

The first and the second terms are more complicated. Using the notation $M_3 \equiv M_\lambda^{(3)}$ and $M_4 \equiv M_\lambda^{(4)}$, and performing the change of variable $y \to t = -y(M_3/\sqrt{N})^{1/3}$, we will have in both

$$
\ln \left[\frac{1}{|N^{1/3} M_3^{1/3}|} \int_{-\infty}^{+\infty} \mathrm{d}t \exp\left(t^3 - \frac{M_4}{N^{1/3} M_3^{4/3}} t^4 \right) \right] . \tag{3.19}
$$

Note that the coefficient of t^4 in (3.19) must be positive for the convergence of the integral; in the last section, in (5.10), we will show that this condition is fulfilled. In practice, the effect of the quartic term is to cut the integral in $\mathrm{d}t$ at the value

$$
t_0 = \frac{N^{1/3} M_3^{4/3}}{M_4} . \tag{3.20}
$$

For smaller values of t the exponential is dominated by the t^3 term, and we have

$$
\int_{-\infty}^{+\infty} \mathrm{e}^{t^3 - t^4/t_0}\, \mathrm{d}t \simeq \int_{-\infty}^{t_0} \mathrm{e}^{t^3}\, \mathrm{d}t = \frac{\mathrm{e}^{t_0^3}}{3t_0^2} \left(1 - \frac{2}{3t_0^3} + \mathcal{O}(t_0^{-6}) \right) \qquad \text{for } t_0 \gg 1 \tag{3.21}
$$

and a good estimate of the (3.19) is given by

$$\ln\left[\frac{1}{|N^{1/3}M_3^{1/3}|}\frac{M_4^2}{3N^{2/3}M_3^{8/3}}\exp\left(\frac{NM_3^4}{M_4^3}\right)\right]=\ln\left[\left|\frac{M_4^2}{3N^{1/2}M_3^3}\right|\exp\left(\frac{NM_3^4}{M_4^3}\right)\right]. \tag{3.22}$$

Substituting (3.22) in (3.17) gives, for the free energy density,

$$\begin{aligned}-\beta(f-f_0)=&\frac{1}{N}\sum_{\lambda=0}\left(\frac{NM_3^4}{M_4^3}+\ln\left|\frac{M_4^2}{3NM_3^3}\right|\right)\\&+\frac{1}{N}\int_0^{\lambda_{cr}}\mathrm{d}\lambda\,\rho(\lambda)\left(\frac{NM_3^4}{M_4^3}+\ln\left|\frac{M_4^2}{3NM_3^3}\right|\right)\\&+\frac{1}{2N}\int_{\lambda_{cr}}\mathrm{d}\lambda\,\rho(\lambda)\ln\frac{\pi}{\lambda N}.\end{aligned} \tag{3.23}$$

4. The critical exponents

Now, to proceed with our computation, we must study the behaviour of the quantities λ_{cr}, ρ, M_3 and M_4 introduced above.

To do this, we will introduce and calculate four critical exponents that will describe their behaviour in the $N\to\infty$ limit.

Let us suppose that, for $N\gg 1$,

$$\lambda_{cr}\sim N^{-\gamma} \tag{4.1}$$

and that consequently, for small λ,

$$\rho(\lambda)\sim\lambda^{\delta}\qquad M_4\sim\lambda^{\alpha}\qquad M_3\sim\lambda^{\beta}. \tag{4.2}$$

Substituting these exponents in the Gaussian term (3.18), we find the contribution of the fluctuations with the bigger eigenvalues to the free energy density

$$\frac{N^{-[1+\gamma(\delta+1)]}}{2(\delta+1)^2}\left[-(\gamma+1/2)(\delta+1)\ln N-1+(\delta+1)\ln\pi\right]. \tag{4.3}$$

This result is valid if $\delta\neq-1$, but the integral (3.18) is always convergent.

The second term, the integral over small eigenvalues in (3.17), gives

$$\begin{aligned}\frac{1}{N}\int_0^{N^{-\gamma}}\mathrm{d}\lambda\,\lambda^{\delta}\left[-N\lambda^{4\beta-3\alpha}+\ln\frac{\lambda^{2\alpha-3\beta}}{3N}\right]\\=\frac{(6\beta\gamma-4\alpha\gamma-1)\ln N+6\beta-4\alpha-2(1+\delta)\ln 3}{2(1+\delta)^2N^{\gamma(1+\delta)+1}}\\-\frac{N^{-(\gamma+4\beta\gamma+\gamma\delta-3\alpha\gamma-2)}}{(1+4\beta-3\alpha+\delta)}\end{aligned} \tag{4.4}$$

if

$$\delta > -1 \qquad \text{and} \qquad 4\beta - 3\alpha + \delta > -1\,. \tag{4.5}$$

Otherwise, it is divergent.

Finally, the contribution of the zero modes remains:

$$\sum_{\lambda=0}\left(\frac{M_3^4}{M_4^3} + \frac{1}{N}\ln\left|\frac{M_4^2}{3NM_3^3}\right|\right)\,. \tag{4.6}$$

In the next section we will calculate these exponents, so obtaining separately the contributions of the different λ regions.

5. Results

5.1. *The exponent* β

Let us study the behaviour of

$$M_\lambda^{(3)} \equiv \delta_3\mathcal{F}[f_\lambda] = \tfrac{1}{2}\int_0^1 \mathrm{d}x\left(\left(q_{\mathrm{CM}}(x) - x/3\right)f_\lambda(x)^3 - f_\lambda(x)\int_0^x f_\lambda(y)^2\mathrm{d}y\right) \tag{5.1}$$

for small values of λ.

We begin by considing (5.1) for the eigenvectors of the family $f_\lambda \neq 0$ only if $0 \leqslant x \leqslant 3q_1$; for small eigenvalues, the (3.8), (3.9) and (3.10) become

$$\omega_\lambda = \sqrt{\frac{2}{3\lambda}} \qquad a_\lambda = 3\omega_\lambda b_\lambda \to \sqrt{\frac{2}{3q_1}} \tag{5.2}$$

$$b_\lambda \sim \sqrt{\frac{2}{27q_1\omega_\lambda^2}} = \sqrt{\frac{\lambda}{9q_1}} \to 0 \qquad \tan(3\omega_\lambda q_1) \simeq \frac{6q_1 - 2}{\sqrt{6\lambda}} \tag{5.3}$$

so that we can neglect the term $b_\lambda \sin\omega_\lambda x$ in the eigenfunctions and consider

$$f_\lambda \simeq a_\lambda \cos\omega_\lambda x \qquad \text{when} \quad \lambda \to 0\,. \tag{5.4}$$

For eigenfunctions of this type, (5.1) gives

$$\begin{aligned}\delta_3\mathcal{F}[f_\lambda] &= \frac{\lambda}{12q_1}\sqrt{\frac{2}{3q_1}}[2 - 3\cos(3q_1\omega_\lambda) + \cos(3q_1\omega_\lambda)^3 - 9q_1\omega_\lambda\sin(3q_1\omega_\lambda)] \\ &\to \frac{\lambda}{12q_1}\sqrt{\frac{2}{3q_1}}[-9q_1\omega_\lambda\sin(3q_1\omega_\lambda)] = -\frac{\sqrt{\lambda}}{2q_1^{3/2}}\sin(3q_1\omega_\lambda)\end{aligned} \tag{5.5}$$

from which we have the first result:

$$\beta = \tfrac{1}{2}\,. \tag{5.6}$$

For the zero modes, we have

$$\delta_3\mathcal{F}[f_\lambda] = \frac{13\sqrt{2(1-3q_1)}\cos k\pi}{18k\pi} \tag{5.7}$$

for the eigenfunctions of the type

$$f_k(x) = \sqrt{\frac{2}{1-3q_1}}\sin\left(\frac{x-3q_1}{1-3q_1}k\pi\right)\Theta(x-3q_1)$$

and

$$\delta_3\mathcal{F}[f_\lambda] = \frac{16\sqrt{2(1-3q_1)}(1-\cos k\pi)}{27k^2\pi^2} \tag{5.8}$$

for the eigenfunctions of the type

$$f_k(x) = \sqrt{\frac{2}{1-3q_1}}\cos\left(\frac{x-3q_1}{1-3q_1}k\pi\right)\Theta(x-3q_1)$$

where $\Theta(y)$ is the step function.

5.2. *The exponent* α

Now we consider

$$M_\lambda^{(4)} \equiv \delta_4\mathcal{F}[f_\lambda] = \frac{1}{2}\int_0^1 \mathrm{d}x\,\frac{f_\lambda(x)^4}{4} \tag{5.9}$$

for small λ values.

Again, we consider first the family $f_\lambda(x) \neq 0$ when $0 \leqslant x \leqslant 3q_1$, that is $f_\lambda \simeq a_\lambda \cos\omega_\lambda x$ as we have seen in (5.4). We have

$$\delta_4\mathcal{F}(f_\lambda) = \frac{9q_1\omega_\lambda + 3\cos(3q_1\omega_\lambda)\sin(3q_1\omega_\lambda) + 2\cos(3q_1\omega_\lambda)^3\sin(3q_1\omega_\lambda)}{144\,q_1^2\,\omega_\lambda} \to \frac{1}{16q_1} \tag{5.10}$$

from which we obtain

$$\alpha = 0 \tag{5.11}$$

and $M_4 > 0$, as was necessary for the convergence of integral (3.19).

For the second family, $f_\lambda(x) \neq 0$ when $3q_1 \leqslant x \leqslant 1$ (the zero modes), we have two identical contributions from the sines and the cosines, namely

$$\delta_4\mathcal{F}(f_\lambda) = \frac{3}{16-48q_1} \tag{5.12}$$

for both.

5.3. *The exponent* γ

λ_{cr} is the value in which the quadratic and the cubic part of the exponential in (3.16) are of the same order. To find it, we consider both terms in (3.16) (to find λ_{cr} we can neglect the quartic term):

$$\int dy \exp\left[-\lambda y^2 + \frac{M_3}{\sqrt{N}} y^3\right] = \int \frac{dz}{\sqrt{\lambda}} \exp\left[-z^2 + \frac{M_3}{\lambda^{3/2}\sqrt{N}} z^3\right] . \tag{5.13}$$

The integrand can be consider purely Gaussian when

$$\frac{M_3}{\lambda^{3/2}\sqrt{N}} \ll 1 \tag{5.14}$$

but $M_3 \sim \lambda^{1/2}$ for small λ, so that (5.14) becomes

$$\lambda \gg \lambda_{cr} \sim N^{-1/2} . \tag{5.15}$$

That means

$$\gamma = \tfrac{1}{2} . \tag{5.16}$$

5.4. *The exponent* δ

The eigenvalue equation (5.3) can be written

$$\tan(3\omega_\lambda q_1) = 3\omega_\lambda q_1 - \omega_\lambda \tag{5.17}$$

which, with $x = 3\omega_\lambda q_1$, becomes

$$\tan x = (1 - r)x \tag{5.18}$$

where $r = (3q_1)^{-1}$. The solutions x_k of (5.18) for small λ (i.e. large x) can be found putting $x_k = \pi/2 + k\pi - \epsilon_k$ and solving in ϵ_k to have

$$\epsilon_k = \frac{1}{(1-r)k\pi} + \mathcal{O}(k^{-2}) . \tag{5.19}$$

Returning to the x_k and the $\lambda_k = 2/(3\omega_k^2)$, we have

$$\lambda_k = \frac{6q_1^2}{k^2\pi^2}\left(1 - \frac{1}{k} + \mathcal{O}(k^{-2})\right) . \tag{5.20}$$

To establish the behaviour of $\rho(\lambda)$, we must find the number of eigenvalues between λ and $\lambda + \delta\lambda$ when $\lambda \to 0$. Let k and k' be two integers such that

$$\frac{6q_1^2}{k^2\pi^2} = \lambda \qquad \text{and} \qquad \frac{6q_1^2}{k'^2\pi^2} = \lambda + \delta\lambda . \tag{5.21}$$

The number of eigenvalues in the considered interval will be

$$k - k' = \sqrt{\frac{6q_0^2}{\pi^2\lambda}} - \sqrt{\frac{6q_0^2}{\pi^2(\lambda+\delta\lambda)}} = \sqrt{\frac{6q_0^2}{\pi^2\lambda}}\left[1 - \left(1 + \frac{\delta\lambda}{\lambda}\right)^{-1/2}\right] \simeq \sqrt{\frac{6q_0^2}{\pi^2\lambda}}\,\frac{\delta\lambda}{2\lambda} \tag{5.22}$$

so that the eigenvalue density near $\lambda = 0$ is

$$\rho(\lambda) = \frac{k - k'}{\delta\lambda} \simeq \sqrt{\frac{3q_0^2}{2\pi^2\lambda^3}} \tag{5.23}$$

implying that

$$\delta = -\tfrac{3}{2} . \tag{5.24}$$

5.5. *Final result: correction to the free energy density*

Now that we have found all the critical exponents, we must verify the conditions (4.5) for the convergence of the small eigenvalues' contributions.

Substituting $\alpha = 0$ and $\beta = \frac{1}{2}$ in (4.5), we have the unique convergence condition $-3 < \delta < -1$, so that we can accept the value $\delta = -\frac{3}{2}$ found in (5.24) and the first corrections in N to the free energy density coming from positive eigenvalues are

$$-\beta(f - f_0) = \frac{N^{-3/4}}{6}(-63 \ln N + 20 - 6 \ln \pi + 12 \ln 3) + o(N^{-3/4}). \qquad (5.25)$$

As we had anticipated, we find that the main correction is of the order $N^{-3/4} \ln N$: the value $\frac{3}{4}$ comes from the Gaussian approximation but indicates the presence of logarithms and non-integer exponents in the exact expansion.

Unfortunately, we also find divergent contributes from the zero modes: they give a term of the type

$$c(1 - 3q_1)^5 \qquad \text{with} \quad c = 1.834\,474\,033\ldots \qquad (5.26)$$

that is of the same order of magnitude as f_0 and, worse still, another term of the type

$$c' \frac{\ln N}{N} \sum_{k=2}^{\infty} \frac{1}{\ln k^3} \qquad (5.27)$$

that is divergent.

The reason for this divergence is the first approximation made in section 3, when we considered only the longitudinal eigenvalues: considering all the fluctuations [4] we find that zero modes also exist in the other families of eigenvalues and that their contributions cancel this divergence. This fact will be shown in detail in a future publication.

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